On the matched asymptotic solution of the Troesch problem (*)

R. C. Y. Chin (**)

ABSTRACT

We present a composite expansion solution to the Troesch problem using the method of matched asymptotic expansions. Our solution is uniformly valid for all $n>0$ and $y \geqslant 0$ and, thus, subsumes Anglesio and Troesch's approximate solution for large n.

1. INTRODUCTION

Recently, Anglesio and Troesch [1] presented a derivation of an approximation for large $n>0$ to $y'(0)$ and $y(x)$ for the problem

$$y'' = n \sinh ny$$

with boundary conditions

$$y(0) = 0 \quad \text{and} \quad y(1) = 1.$$

This problem, called the Troesch problem [2], was proposed as a model to illustrate the difficulties encountered in the numerical solution of a two-point boundary value problem with a severe boundary layer. This nonlinear differential equation traces its origin to the problem of the confinement of a plasma column by radiation pressure [2] and occurs also in the theory of gas porous electrodes [3] [4].

In this note, we present a composite expansion solution to the Troesch problem using the method of matched asymptotic expansions. The composite asymptotic solution is uniformly valid for all $n>0$ and for $y \geqslant 0$. Anglesio and Troesch's approximate solution corresponds to the outer solution of our composite asymptotic solution. Our result extends and subsumes their approximation to $y \geqslant 0$ for any $n>0$. Moreover, this problem presents an excellent pedagogic example for the application of the matching principles in the method of matched asymptotic expansions and for the use of differential inequalities in estimating the solution behavior to nonlinear boundary value problems.

Closed form solutions in terms of elliptic functions are known, but they don't give much insight into the behavior of the solution. Therefore, they are included only for the sake of completeness. In this context, our result also provides an asymptotic expansion of the incomplete elliptic integral of the first kind $F(\phi \backslash \alpha)$ as $\phi, \alpha \to \pi/2$.

2. REPRESENTATION OF THE SOLUTION

Following the usual procedure in the solution of second order autonomous differential equations, we obtain a first integral of the equation :

$$y'^2(x) = y'^2(0) + 4 \sinh^2 (ny/2). \tag{1}$$

Defining $\beta = y'(0)/2$, and introducing the new variable ζ,

$$\zeta(y) = \sinh (ny/2)$$

we rewrite (1) as

$$n \frac{dx}{d\zeta} \frac{1}{\sqrt{(1+\zeta^2)(\zeta^2+\beta^2)}} \tag{2}$$

and the boundary condition $y(1) = 1$ becomes

$$\zeta(1) = \sinh (n/2).$$

Integrating (2), we have

$$nx = \int_0^{\sinh (ny/2)} \frac{d\zeta}{\sqrt{(1+\zeta^2)(\zeta^2+\beta^2)}}$$

The last integral may be rewritten using a formula from Gradshteyn and Ryzhik [5] in terms of an incomplete elliptic integral of the first kind,

$$nx = F(\alpha, q)$$

where $\alpha = \tan^{-1}\left[\frac{1}{\beta} \sinh (ny/2)\right]$

$$q = \sqrt{1-\beta^2}.$$

In the more familiar notation of Abramowitz and Stegun [6], we have

$$\phi = \tan^{-1}\left[\frac{1}{\beta} \sinh (ny/2)\right],$$

$$\alpha = \sin^{-1}\sqrt{1-\beta^2},$$

$$\text{and } nx = F(\phi \backslash \alpha). \tag{3}$$

Roberts and Shipman [7] give the solution in terms of the Jacobian elliptic function "sc(u/m)" :

(*) Work performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore National Laboratory under contract number W-7405-ENG-48.

(**) R. C. Y. Chin, Lawrence Livermore National Laboratory, Livermore, California 94550.

$$y(x;n) = \frac{2}{n} \sinh^{-1} \left[\beta\, sc(nx| 1-\beta^2)\right]$$

To complete the solution of the Troesch problem, we use the boundary condition y(1) = 1 to obtain

$$n = F\left[\tan^{-1}(\frac{1}{\beta} \sinh n/2)\backslash \sin^{-1}\sqrt{1-\beta^2}\right]. \quad (4)$$

Equation (4) gives a relation between $\beta = y'(0)/2$ and n. A bound on $\beta = y'(0)/2$ can be easily derived using differential inequalities [8] : It is readily seen from fig. 1 that

$$ny < \sinh ny < (\sinh n)\, y \quad (5)$$

for $0 \leqslant y \leqslant 1$ and $n > 0$.

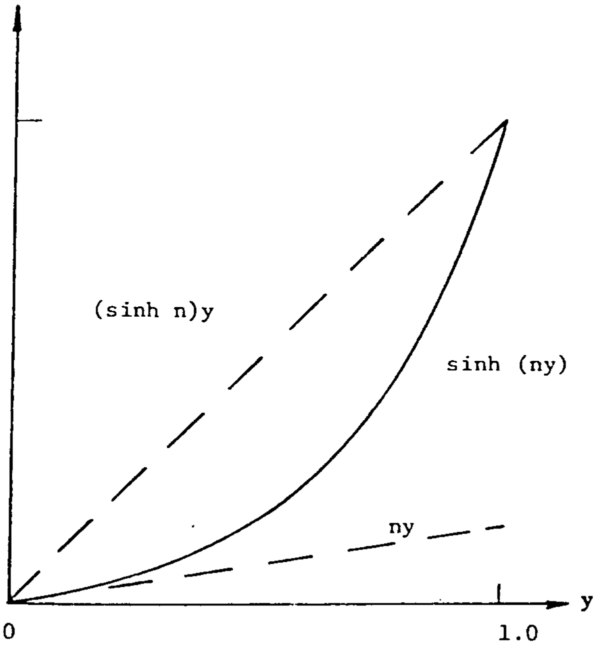

Fig. 1.

Let u_i be the solution of the following boundary value problem

$$u_i'' = \alpha_i u_i$$

$$\alpha_1 = n^2, \alpha_2 = n \sinh n$$

with boundary conditions, $u_i(o) = 0$ and $u_i(1) = 1$.
From the inequality (5), we have

$$u_2(x) < y(x) < u_1(x)$$

where $u_1(x)$ is the upper solution and $u_2(x)$ is the lower solution. It follows immediately that

$$\frac{(n \sinh n)^{1/2}}{2 \sinh (n \sinh n)^{1/2}} < \beta < \frac{n}{2 \sinh \hat{n}}.$$

As $\sinh n \geqslant n$, then

$$\beta < \frac{1}{2}$$

for all $n > 0$ (as noted in [2]). We will make use of this fact in developing a solution to equation (2).

3. MATCHED ASYMPTOTIC EXPANSION

In this section, we develop a solution to equation (2) using the method of matched asymptotic expansions [9], [10], [11]. An inner and an outer solution are developed as power series in β. These solutions are then matched in an overlapping domain according to the matching principles of Van Dyke [11]. For completeness, we quote the asymptotic matching principle: "The m-term inner expansion of (the n-term outer expansion) = the n-term outer expansion of (the m-term inner expansion)".
To develop an inner solution, we begin by introducing the inner variable n

$$\zeta = \beta\eta, \ 0 \leqslant \eta < 1$$

Then (2) becomes

$$n \frac{dx}{d\eta} = \frac{1}{\sqrt{1+\eta^2}\ \sqrt{1+\beta^2\eta^2}}$$

For $(\beta\eta)^2 < 1$, $(1+\beta^2\eta^2)^{-1/2}$ may be developed as a convergent power series.
Similarly, we expand nx^i, the inner solution, as a power series in β^2 :

$$nx^i(\eta;\beta) = \sum_{k=0}^{\infty} u_k(\eta)\, \beta^{2k} \quad (6)$$

Since $\beta < 1/2$ this series is convergent if $u_k(\eta)$ k = 0, ..., are uniformly bounded. Substituting the expansions for $(1+\beta^2\eta^2)^{-1/2}$ and for nx^i and collecting equal powers of β^2, we have

$$\frac{du_0}{d\eta} = \frac{1}{\sqrt{1+\eta^2}}$$

$$\frac{du_1}{d\eta} = -\frac{1}{2} n^2/\sqrt{1+\eta^2}$$

$$\frac{du_2}{d\eta} = \frac{3}{8} \eta^4/\sqrt{1+\eta^2}$$

$$\frac{du_k}{d\eta} = a_k \eta^{2k}/\sqrt{1+\eta^2} \qquad k > 2$$

where $a_k = -(1-\frac{1}{2k})\, a_{k-1}$ with $a_0 = 1$.

The initial conditions for u_k, k = 0, 1, 2 ... are

$$u_k(0) = 0$$

since $x(0) = 0$.
Integrating term by term and using the homogeneous initial conditions, we obtain

$$u_0(\eta) = \ln(\eta + w)$$

$$u_1(\eta) = \frac{-1}{4}[\eta w - \ln(\eta + w)]$$

$$u_2(\eta) = \frac{3}{64}[2\eta^3 w - 3\eta w + 3\ln(\eta + w)]$$

where $w = \sqrt{1+\eta^2}$

For the outer solution, we rewrite (2) as

$$n\frac{dx}{d\zeta}=\frac{1}{\zeta\sqrt{1+\zeta^2}\ \sqrt{1+\beta^2/\zeta^2}}$$

A convergent expansion for $(1+\beta^2/\zeta^2)^{-1/2}$ occurs if $\beta<\zeta$

Since $\beta<1/2$, there exists an overlapping domain R, $R=\{\zeta\,|\,\beta<\zeta<1\}$ in which both the inner and outer solutions have valid expansions.

Before proceeding to the construction of the outer solution, we investigate the inner solution behavior in the overlapping domain to assess the functional form of the outer solution. This is done by constructing the outer expansion of the inner solution, i.e., writing the inner variable, $\eta=\zeta/\beta$ and expanding the resultant expression in β^2. In so doing, we obtain

$$nx^i(\zeta;\beta)=u_0(\zeta/\beta)+u_1(\zeta/\beta)\,\beta^2+0\,(\beta^4)$$

$$=\ln\frac{2\zeta}{\beta}-\frac{1}{4}\zeta^2+\beta^2\left[\frac{1}{4\zeta^2}-\frac{1}{8}\right]+\frac{1}{4}\beta^2\ln\frac{2\zeta}{\beta}+\ldots\ldots \qquad (7)$$

Note the occurence of the terms $\beta^{2k}\ln\beta$, k=0,1,2..... Since the outer solution must be matched onto the inner solution in the overlapping domain, this suggests that the outer solution takes on the form

$$nx^0(\zeta;\beta)=\sum_{k=0}^{\infty}\beta^{2k}v_k(\zeta)+\sum_{k=0}^{\infty}\bar v_k\,\beta^{2k}\ln\beta$$

Following a procedure similar to that used for constructing the inner solution, we obtain

$$v_0(\zeta)=\frac{1}{2}\ln\left(\frac{Z-1}{Z+1}\right)+\gamma_0,$$

$$v_1(\zeta)=\frac{1}{4}\left[Z/\zeta^2+\frac{1}{2}\ln\left(\frac{Z-1}{Z+1}\right)\right]+\gamma_1,$$

$$v_2(\zeta)=\frac{3}{64}\left[\frac{3}{2}\ln\left(\frac{Z-1}{Z+1}\right)+\frac{3Z}{\zeta^2}-\frac{2Z}{\zeta^2}\right]+\gamma_2,$$

where $Z=\sqrt{1+\zeta^2}$

The constants γ_i i = 0,1,2,... and $\bar v_k$ k = 0,1,2,... are obtained by matching term by term the inner expansion of the outer solution to the outer expansion of the inner solution.

The results of the matching are

$$\bar v_0=-1,\ \bar v_1=\frac{-1}{4},\ \ldots\ldots$$

$$\gamma_0=\ln 4,\ \gamma_1=\frac{-1}{4}(1-\ln 4),\ \ldots\ldots$$

The outer solution valid for $\zeta>\beta$ becomes

$$nx^0(\zeta;\beta)=v_0(\zeta)+\beta^2v_1(\zeta)+\bar v_0\ln\beta+\bar v_1\beta^2\ln\beta+0(\beta^4\ln\beta)$$

$$=\ln 4/\beta+\frac{1}{2}\ln\frac{Z-1}{Z+1}+\frac{1}{4}\beta^2\ln(4/\beta)$$

$$+\frac{\beta^2}{4}\left[Z/\zeta^2+\frac{1}{2}\ln\frac{Z-1}{Z+1}-1\right]+0\,(\beta^4\ln\beta) \qquad (8)$$

with $\zeta=\sinh(ny/2)$ and $Z=\sqrt{1+\zeta^2}$

Equation (8) is an expansion in β^{2k} and $\beta^{2k}\ln\beta$. By regrouping the terms, we may rewrite (8) as

$$nx^0(\zeta;\beta)=\ln\frac{4}{\beta}\sqrt{\frac{Z-1}{Z+1}}+\frac{\beta^2}{4}\left[\ln\frac{4}{\beta}\sqrt{\frac{Z-1}{Z+1}}+\frac{Z}{\zeta^2}-1\right]+0\,(\beta^4\ln\beta). \qquad (9)$$

Having the inner and outer solutions, we form an uniformly valid composite solution by summing the inner and outer solution equations (6) and (8) and by subtracting the outer expansion of the inner solution, equation (7). This gives a composite expansion which is uniformly valid for $y\geqslant 0$:

$$nx(\zeta;\beta)=\ln\left(\frac{\zeta}{\beta}+w\right)-\frac{\beta^2}{4}\left[\zeta w/\beta-\ln(\zeta/\beta+w)\right]$$

$$+\ln\frac{4}{\beta}\sqrt{\frac{Z-1}{Z+1}}+\frac{\beta^2}{4}\left[\frac{Z}{\zeta^2}+\ln\frac{4}{\beta}\sqrt{\frac{Z-1}{Z+1}}-1\right]$$

$$-\ln(2\zeta/\beta)+\frac{\zeta^2}{4}-\frac{\beta^2}{4}\left[\frac{1}{\zeta^2}-\frac{1}{2}\right]-\frac{1}{4}\beta^2\ln(2\zeta/\beta)$$

$$+0\,(\beta^4\ln\beta) \qquad (10)$$

with $\zeta=\sinh(ny/2)$, $w=\sqrt{1+\zeta^2/\beta^2}$, $Z=\sqrt{1+\zeta^2}$.

Higher accuracy may be obtained by performing the matching of the inner and outer solutions to higher orders in β.

We plot respectively for n = 5 in figs. 2-4 the inner solution, the outer solution and both the inner and outer solutions. Fig. 4 clearly shows the overlapping domain and the matching of the inner and outer solutions.

This result, equation (10) is new and is uniformly valid for all $n>0$ by virtue of $\beta(n)<1/2$ as provided by the differential inequality estimate. The derivation is complete and self-contained, i.e., a study of the literature is not required. Moreover, the method is general and not specific to this particularly simple problem. To show that our equation (10) contains Anglesio and Troesch's approximate solution, we rewrite the outer solution, equation (8) in terms of sinh (ny/2) and identify

$$J(ny/2;\beta)=\int_{ny/2}^{\infty}\frac{du}{\sqrt{\beta^2+\sinh^2 u}}=\sum_{k=0}^{\infty}\beta^{2k}v_k(\zeta)$$

and $2J(u_0;\beta)=\sum_{k=0}^{\infty}\beta^{2k}\left[\gamma_k+\bar v_k\ln\beta\right]$

where $u_0=\sinh^{-1}\sqrt{\beta}$.

4. DISCUSSION

To calculate $y'(0)$ for large n, we have

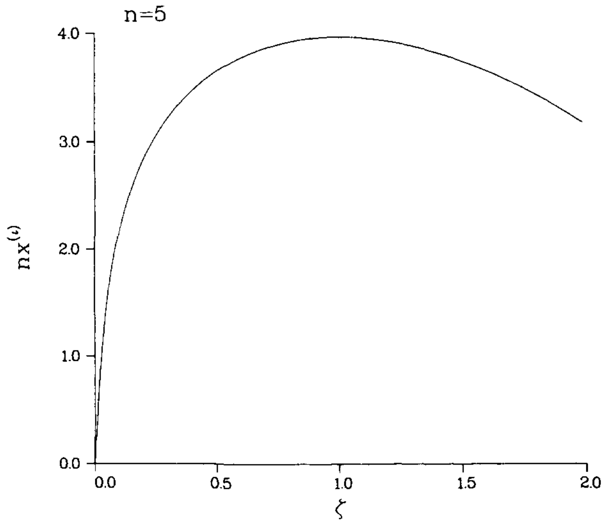

Fig. 2.

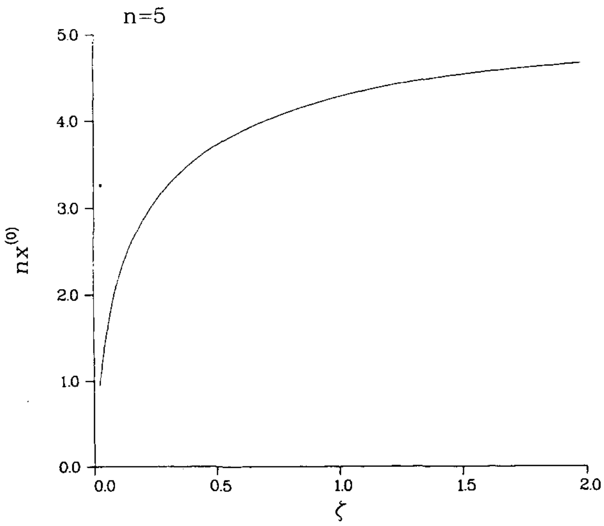

Fig. 3.

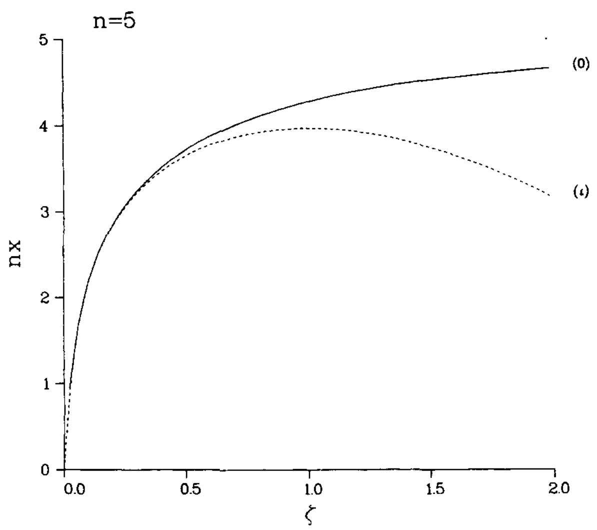

Fig. 4.

$$\beta < ne^{-n} < \frac{n}{2 \sinh n} \ll 1$$

and $(1) = \sinh (n/2) \gg 1$.

This implies that the outer solution $x^0(\zeta;\beta)$ may be used to evaluate $y'(0)$. Neglecting terms higher than $0\,(\beta^2 \ln\beta)$ in equation (8), we obtain

$$n = \ln \frac{4}{\beta} \sqrt{\frac{Z-1}{Z+1}} = \ln \frac{4}{\beta} \sqrt{\frac{\sinh n/2 - 1}{\sinh n/2 + 1}} = \ln \frac{4}{\beta} \tanh (n/4)$$

and

$$y_0' (0) = 8e^{-n} \tanh (n/4) \tag{11}$$

Keeping terms to $0(\beta^4 \ln\beta)$, we have as in Anglesio and Troesch

$$y'(0) = y_0'(0) \left[1 + \left[\frac{y_0'(0)}{4} \right]^2 \left[n - 1 + \frac{\cosh (n/2)}{\sinh^2 (n/2)} \right] \right] \tag{12}$$

It is remarkable that the approximation, equation (10) has higher accuracy $0(ne^{-3n})$, as indicated by equation (11), than the anticipated $0(ne^{-2n})$.

ACKNOWLEDGEMENT

The author would like to thank the referee for a diligent review and for suggestions on improving the manuscript; G. W. Hedstrom and R. L. Pexton for many helpful discussions; and Mrs. Lila Chase for providing the graphics.

REFERENCES

1. ANGLESIO J. D. and TROESCH B. A. : "High precision results for a two-point boundary value problem", J. Comp. Appl. Math., 6 (1980), 99-103.
2. TROESCH B. A. : "A simple approach to a two-point boundary value problem", J. Comp. Phys., 21 (1976), 279-290.
3. MARKIN V. S., CHERNENKO A. A., CHIZMADZHEV YU A. and CHIRKOV YU G. : "Aspects of the theory of gas porous electrodes", *Fuel Cells, their electrochemical kinetics,* BAGOTSKII, V. S. and VASIL'EV, YU. B., Ed. Consultants Bureau, N.Y. (1966), 21-33.
4. GIDASPOW D. and BAKER B. S. : "A model for discharge of storage batteries", J. Electrochem. Soc., 120 (1973), 1005-1010.
5. GRADSHTEYN I. S. and RYZHIK I. M. : *Table of integrals, series and products,* Academic Press, N. Y. (1965), 245.
6. ABRAMOWITZ M. and STEGUN I. A. : *Handbook of mathematical functions,* Nat. Bur. Standards, Dept. AMS 55, (1964), 596.
7. ROBERTS S. M. and SHIPMAN J. S. : "On the closed form solution of Troesch's problem", J. Comp. Phys. 21 (1976), 291-304.
8. BERNFELD S. R. and LAKSHMIKANTHAM V. : *An introduction to nonlinear boundary value problems,* Academic Press, N. Y. (1974), Chap. 1.
9. O'MALLEY R. E. JR. : *Introduction to singular perturbations,* Academic Press, New York (1974), Chaps. 1 and 5.

10. NAYFEH A. H. : *Perturbation methods,* John Wiley and Sons, New York (1973), Chap. 4.

11. VAN DYKE M. : *Perturbation methods in fluid mechanics,* The Parabolic Press, Stanford, California (1975), Chap 5.